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Viscosity solutions of the p-Laplacian diffusion equation

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1. Introduction

In this note we consider the Cauchy problem of the p-Laplacian diffusion equation of the form

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in} \quad Q_T := (0,T) \times \boldsymbol{R}^N, \tag{1.1}$$

$$u(0,x) = a(x) \quad \text{on} \quad \boldsymbol{R}^N, \tag{1.2}$$

where $u\colon Q_T \to \boldsymbol{R}$ is an unknown function, $a(x)$ is continuous, $T > 0$ and $p > 1$. Here $u_t = \partial u/\partial t$ and ∇u denote, respectively, the time derivative of u and the gradient of u in space variables. This equation is well known and studied by many authors. The p-Laplacian diffusion equation is degenarate parabolic. So we cannot expect to get classical solutions. Usually, to study this equation many authors use usual weak solutions defined in distribution sense, since the p-Laplacian diffusion equation has the divergence structure. However, here we introduce a notion of viscosity solutions for the p-Laplacian diffusion equation. A notion of viscosity solutions was introduced by Crandall and Lions. We refer to a nice review paper by Crandall, Ishii and Lions [CIL]. The definition does not require the divergence structure of equations. This is an our advantage. Our purpose of this note is to introduce a notion of viscosity solutions for singular degenerate parabolic equations including the p-Laplacian diffusion equation with $p > 1$. Then we show a comparison theorem and the unique existence theorem.

Before to state a notion of viscosity solutions of (1.1), we would like to write equations in a general form. We consider singular degenerate parabolic equations of the form

$$u_t + F(\nabla u, \nabla^2 u) = 0 \quad \text{in} \quad Q_T, \tag{1.3}$$

where $F = F(q,X)$ is a given function. Here $\nabla^2 u$ denotes the Hessian of u in space variables. The function $F = F(q,X)$ needs not to be bounded around $q = 0$ even for fixed X and needs not to be *geometric* in the sense of Chen, Giga and Goto [CGG], i.e.,

$$F(\lambda q, \lambda X + \mu q \otimes q) = \lambda F(q,X) \quad \text{for all} \quad \lambda > 0, \mu \in \boldsymbol{R}, q \in \boldsymbol{R}^N \setminus \{0\}, X \in \boldsymbol{S}^N,$$

where $\boldsymbol{S}^N$ denotes the space of all $N \times N$ real symmetric matrices.

For the p-Laplacian diffusion equation (1.1) we give $F(q,X)$ of the form

$$F(q,X) = -|q|^{p-2} \operatorname{trace}\left\{ \left(I + (p-2)\frac{q \otimes q}{|q|^2}\right) X \right\}, \tag{1.4}$$

where $\otimes$ denotes the tensor product. Note that when $1 < p < 2$ the value of $F(q, X)$ in (1.4) is irrelevant for $q = 0$. For such singular function F Chen, Giga and Goto [CGG] introduced a notion of viscosity solutions. Independently, for special F which comes from the mean curvature flow equation Evans and Spruck [ES] introduced a notion of viscosity solutions. The function $F(q, X)$ is not continuous for $q = 0$ but $F^*(0, O)$ and $F_*(0, O)$ are bounded. Here F^* and F_* denote upper semicontinuous envelope of F and lower semicontinuous envelope of F, respectively (cf. [CGG]). Then Ishii and Souganidis [IS] introduced a notion of viscosity solutions for F which satisfies $F^*(0, O)$ and $F_*(0, O)$ are not bounded. They assume that F is *geometric* in the sense of [CGG]. In the same time Goto [G] studied a problem under similar situations of [IS]. He used a notion of viscosity solutions as in [CGG] and overcame the problem using another technique. Our notion of solutions of (1.3) is a natural extension of the paper by Ishii and Souganidis [IS]. We do not assume that F is *geometric* in the sense of [CGG]. So we can treat the p-Laplacian diffusion equation.

A comparison principle, which is a natural extension of the paper by Ishii and Souganidis [IS], for (1.3) was established by the author and K. Sato [OS]. Once the comparison principle for (1.3) was proved, we can construct the unique global-in-time viscosity solution of (1.3)-(1.2). Moreover, we see that the solution is bounded, uniformly continuous in $[0, T) \times \boldsymbol{R}^N$ provided that the initial data is bounded, uniformly continuous on $\boldsymbol{R}^N$ (cf. [OS]).

Here we shall write a little bit generalized equation of (1.1)

$$u_t - |\nabla u|^{p-2} \operatorname{trace}\left\{ (I + (p' - 2)\frac{\nabla u \otimes \nabla u}{|\nabla u|^2})\nabla^2 u \right\} = 0 \quad \text{in} \quad Q_T, \tag{1.5}$$

where $p' \geq 1$ and $p > 1$. For this equation

$$F(q, X) = -|q|^{p-2} \operatorname{trace}\left\{ (I + (p' - 2)\frac{q \otimes q}{|q|^2})X \right\}. \tag{1.6}$$

The equation (1.5) has interesting examples.

Example 1. If $p = p'$ then (1.5) is nothing but the p-Laplacian diffusion equation (1.1)

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in} \quad Q_T.$$

Our unique existence theorem has already been known by interpreting solutions as usual weak solutions. However, the proof of the continuity of such a weak solution needs many procedures, since it was done by using the Harnak inequality and many a priori estimates. For details, we refer to the book by DiBenedetto [D]. Our procedures are based on Perron's method, so the proof is simpler than that of usual one.

Note that the equation (1.1) is not geometric.

Example 2. If $p = 2$ and $p' = 1$ then (1.5) is the level set mean curvature flow equation

$$u_t - |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0 \quad \text{in} \quad Q_T. \tag{1.7}$$

This equation was initially studied by Chen, Giga and Goto [CGG] and Evans and Spruck [ES]. They established the comparison principle and proved the unique existence theorem of (1.7)-(1.2), independently. In [CGG] they consider more general equations (1.3). To establish the comparison principle they assume $F = F(q, X)$ can be extended continuously at $(q, X) = (0, O)$, i.e., $-\infty < F_*(0, O) = F^*(0, O) < +\infty$, especially F of (1.7) satisfies $F_*(0, O) = F^*(0, O) = 0$. The equation (1.7) does not have the divergence structure. So the theory of usual weak solution does not apply to (1.7). This situation is different from that of (1.1) and (1.7) is geometric.

Example 3. If $p' = 2$ we have

$$u_t - |\nabla u|^{p-2} \Delta u = 0 \quad \text{in} \quad Q_T. \tag{1.8}$$

This can be regarded as a heat equation with an unbounded coefficient. This is not geometric and does not have the divergence structure.

As in [OS] our results applicable to (1.5) with $p' \geq 1$ and $p > 1$ since we do not require F is geometric or the equation has the divergence structure.

2. Definition of viscosity solutions and a comparison theorem

Here and hereafter we shall study a general equation of form

$$u_t + F(\nabla u, \nabla^2 u) = 0 \quad \text{in} \quad Q_T. \tag{2.1}$$

We list assumptions on $F = F(q, X)$.

(F1) F is continuous in $(\boldsymbol{R}^N \backslash \{0\}) \times \boldsymbol{S}^N$.

(F2) F is degenerate elliptic, i.e.,

$$\text{if} \quad X \geq Y \quad \text{then} \quad F(q, X) \leq F(q, Y) \quad \text{for all} \quad q \in \boldsymbol{R}^N \backslash \{0\}.$$

Remark 2.1. We do not assume

$$-\infty < F_*(0, O) = F^*(0, O) < +\infty$$

to include (1.6) with $1 < p < 2$ and $p' \geq 1$ for which $F_*(0, O) = -\infty$ and $F^*(0, O) = +\infty$.

To define viscosity solutions we have to prepare a class of "test functions". This class is important and a part of test functions as space variable functions.

Definition 2.2. We denote by $\mathcal{F}(F)$ the set of function $f \in C^2[0,\infty)$ which satisfies

$$f(0) = f'(0) = f''(0) = 0, \quad f''(r) > 0 \quad \text{for all} \quad r > 0 \tag{2.2}$$

and

$$\lim_{|x|\to 0, x\neq 0} F\big(\pm \nabla f(|x|), \pm\nabla^2 f(|x|)\big) = 0. \tag{2.3}$$

Remark 2.3. Our definition of $\mathcal{F}(F)$ is an extension of that in [IS]. Actually, if F is geometric then the set $\mathcal{F}(F)$ is the same in [IS].

For F of (1.6) with $p' \geq 1$ we shall write an example $f \in \mathcal{F}(F)$ if it is possible.

(i) If $1 < p < 2$ then $f(r) = r^{1+\sigma}$ with $\sigma > 1/(p-1) > 1$.

(ii) If $p \geq 2$ then $f(r) = r^4$.

(iii) If $p \leq 1$ then $\mathcal{F}(F)$ is empty.

On the other hand, if F is geometric then $\mathcal{F}(F)$ is not empty (cf. [IS]).

We shall define a class of test function so called admissible.

Definition 2.4. A function $\varphi \in C^2(Q_T)$ is *admissible* (in short $\varphi \in \mathcal{A}(F)$) if for any $\hat{z} = (\hat{t}, \hat{x}) \in Q_T$ with $\nabla\varphi(\hat{z}) = 0$, there exist a constant $\delta > 0$, $f \in \mathcal{F}(F)$ and $\omega \in C[0,\infty)$ satisfying $\omega \geq 0$ and $\lim_{r\to 0}\omega(r)/r = 0$ such that

$$|\varphi(z) - \varphi(\hat{z}) - \varphi_t(\hat{z})(t-\hat{t})| \leq f(|x-\hat{x}|) + \omega(|t-\hat{t}|) \tag{2.4}$$

for all $z = (t,x)$ with $|z - \hat{z}| < \delta$.

Now we shall introduce a notion of viscosity solutions of (2.1).

Definition 2.5. Assume that (F1) and (F2) hold and that $\mathcal{F}(F)$ is not empty.

1. A function $u\colon Q_T \to \boldsymbol{R}\cup\{-\infty\}$ is a viscosity subsolution of (2.1) if $u^* < +\infty$ on $\overline{Q_T}$ and for all $\varphi \in \mathcal{A}(F)$ and all local maximum point z of $u^* - \varphi$ in Q_T,

$$\begin{cases} \varphi_t(z) + F(\nabla\varphi(z), \nabla^2\varphi(z)) & \leq 0 \quad \text{if} \quad \nabla\varphi(z) \neq 0, \\ \varphi_t(z) & \leq 0 \quad \text{otherwise.} \end{cases}$$

2. A function $u\colon Q_T \to \boldsymbol{R}\cup\{+\infty\}$ is a viscosity supersolution of (2.1) if $u_* > -\infty$ on $\overline{Q_T}$ and for all $\varphi \in \mathcal{A}(F)$ and all local minimum point z of $u_* - \varphi$ in Q_T,

$$\begin{cases} \varphi_t(z) + F(\nabla\varphi(z), \nabla^2\varphi(z)) & \geq 0 \quad \text{if} \quad \nabla\varphi(z) \geq 0, \\ \varphi_t(z) & \geq 0 \quad \text{otherwise.} \end{cases}$$

3. A function u is called a viscosity solution of (2.1) if u is both a viscosity sub- and super-solution of (2.1).

Before we shall explain a comparison theorem, we need an additional assumption on F.

(F3) (i) $\mathcal{F}(F)$ is not empty. (ii) If $f \in \mathcal{F}(F)$ then $af \in \mathcal{F}(F)$ for all $a > 0$.

Remark 2.6. (i) When $p > 1$ and $p' \geq 1$, F of (1.6) satisfies (F1), (F2) and (F3).

(ii) If F is geomtric, then (F1), (F2) and (F3) hold.

Theorem 2.7. *(Comparison theorem)[OS, Theorem 3.9]. Suppose that F satisfies (F1), (F2) and (F3). Let u and v be upper semicontinuous and lower semicontinuous on $\mathcal{R}_T := [0,T) \times \boldsymbol{R}^N$, respectively. Let u and v be a viscosity sub- and super-solution of (2.1), respectively. Assume that u and v are bounded on $\mathcal{R}_T$. Assume that*

$$\lim_{r\to 0} \sup\{u(z) - v(\zeta); (z,\zeta) \in (\partial_p Q_T \times \mathcal{R}_T) \cup (\mathcal{R}_T \times \partial_p Q_T), |z-\zeta| \le r\} \le 0. \quad (2.5)$$

Then

$$\lim_{r\to 0} \sup\{u(z) - v(\zeta); (z,\zeta) \in \mathcal{R}_T, |z-\zeta| \le r\} \le 0.$$

Especially, $u \le v$ in $\mathcal{R}_T$. Here $\partial_p Q_T := (\{0\} \times \Omega) \cup ([0,T] \times \partial\Omega)$ so called parabolic boundary of Q_T when $Q_T = (0,T) \times \Omega$, where Ω is a domain in $\boldsymbol{R}^N$.

3. Unique existence of solutions

We shall construct a viscosity solution to the Cauchy problem of (2.1)-(1.2). Our construction of solutions is based on Perron's method. The pocedure is the same as in [OS] so we omit the proofs. For details see [OS].

As usual we obtain the following two key propositions. We state them without the proof.

Proposition 3.1. *[OS, Proposition 2.5] Assume that (F1), (F2) and (F3) hold. Let $\mathcal{S}$ be a set of subsolutions of (2.1). We set*

$$u(z) := \sup\{v(z); v \in \mathcal{S}\}, \quad \textit{for all} \quad z \in Q_T.$$

If $u^* < +\infty$ in $\overline{Q_T}$, then u is a subsolution of (2.1).

A similar assertion holds for supersolutions of (2.1).

Proposition 3.2. *[OS, Proposition 2.6] Assume that (F1), (F2) and (F3) hold. Let $\mathcal{S}$ be a set of subsolutions of (2.1). Let ℓ and h be a subsolution and a supersolution of (2.1), respectively. Assume that ℓ and h are locally bounded in Q_T and $\ell \le h$ holds. We set*

$$u(z) := \sup\{v(z); v \in \mathcal{S}, \ell \le v \le h \quad \textit{in} \quad Q_T\}, \quad \textit{for all} \quad z \in Q_T.$$

Then u is a solution of (2.1).

To construct a solution we only have to find a sub- and a super-solution, respectively, which fulfills the hypotheses of Proposition 3.2 and the given initial data $a(x)$. From the degenerate elliptic condition (F2), we have a sufficient condition that a C^2 function to be a super- and a sub-solution, respectively.

Lemma 3.3. *Assume that F satisfies (F1), (F2). Suppose that $\mathcal{F}(F)$ is not empty. If $u \in C^2(Q_T)$ satisfies*

$$\begin{cases} u_t(z) + F(\nabla u(z), \nabla^2 u(z)) & \geq 0 \quad \text{if} \quad \nabla u \neq 0, \\ u_t(z) & \geq 0 \quad \text{otherwise}, \end{cases}$$

$$\left(\text{resp.} \quad \begin{cases} u_t(z) + F(\nabla u(z), \nabla^2 u(z)) & \leq 0 \quad \text{if} \quad \nabla u \neq 0, \\ u_t(z) & \leq 0 \quad \text{otherwise}, \end{cases} \right)$$

then u is a viscosity supersolution (resp. subsolution) of (2.1).

Here we shall write down an outline of construction of a solution of (2.1)-(1.2).
(a) Introduction of $\mathcal{G}$ (a family of C^2 functions).
(b) Construction of C^2 typical subsolutions and supersolutions of (2.1), respectively. These are of form: (function of the time variable)+(function of the space variable) and (function of the space variable)$\in \mathcal{G}$.
(c) Construction of a subsolution and a supersolution of (2.1)-(1.2), respectively. Here we will use Proposition 3.1.
(d) We shall check the hypotheses of Proposition 3.2 and Theorem 2.7 (Comparison theorem).
(e) Finally, we can construct a solution of (2.1)-(1.2) by using Propositon 3.2.

Now we shall carry out all steps.
(a) We introduce a set of C^2 functions $\mathcal{G}$;

$$\mathcal{G} := \{g \in C^2[0,\infty); g(0) = g'(0) = 0, g'(r) > 0 \ (r > 0), \lim_{r \to 0} g(r) = +\infty\}.$$

Remark 3.4. (i) If $g(r) \in \mathcal{G}$ then $g(|x|) \in C^2(\boldsymbol{R}^N)$. A direct calculation yields

$$\nabla^2 g(|x|) = \frac{g'(|x|)}{|x|} I + \left(g''(|x|) - \frac{g'(|x|)}{|x|} \right) \left(\frac{x}{|x|} \otimes \frac{x}{|x|} \right).$$

Although $\nabla^2 g(|x|)$ does not appear to be continuous at $x = 0$, it is regarded as a continuous function. Indeed, $\nabla^2 g(0) = g''(0) I$ holds since $\lim_{r \to 0} g'(r)/r = g''(0)$ by the definition of $\mathcal{G}$.
(ii) If $f(r) \in \mathcal{F}(F)$ then $f(r) \in \mathcal{G}$.
(iii) We may assume that

$$\sup_{r \geq 0} g'(r) < +\infty, \quad \sup_{r \geq 0} g''(r) < +\infty.$$

(b) We observe nice properties of F, which is important to construct a sub- and a super-solution, respectively.

Lemma 3.5. *[OS, Lemma 4.3]. Assume that F satisfies (F1), (F2) and (F3). Then the following properties hold.*
(F4)$_+$ There exists $g \in \mathcal{G}$ such that for each $A > 0$, there exists $B > 0$ that satisfies

$$F\big(\nabla(Ag(|x|)), \nabla^2(Ag(|x|))\big) \geq -B \quad \textit{for all} \quad x \in \boldsymbol{R}^N \backslash \{0\}. \tag{3.1}$$

(F4)$_-$ There exists $g \in \mathcal{G}$ such that for each $A > 0$, there exists $B > 0$ that satisfies

$$F\big(\nabla(-Ag(|x|)), \nabla^2(-Ag(|x|))\big) \leq B \quad \textit{for all} \quad x \in \boldsymbol{R}^N \backslash \{0\}. \tag{3.2}$$

Remark 3.6. For F in (1.6) with $p' \geq 1$ and $1 < p < 2$ we can take a function

$$g(r) = \frac{p-1}{p} r^{\frac{p}{p-1}} \in \mathcal{G}$$

that satisfies (F4)$_\pm$. Note that $g(r)$ is not an element of $\mathcal{F}(F)$. When $p' \geq 1$ and $p > 2$ we take

$$g(r) = r - \arctan(r) \in \mathcal{G}.$$

Then we obtain the following by Lemma 3.3.

Lemma 3.7. *[OS, Lemma 4.4]. Assume that F satisfies (F1), (F2) and (F3). Then $u_+(t, x) := Bt + Ag(|x|)$ and $u_-(t, x) := -Bt - Ag(|x|)$ is a viscosity supersolution and a subsolution of (2.1), respectively, where g, A and B are appeared in (F4)$_+$ and (F4)$_-$.*

(c) Since the equation (2.1) is invariant under the translation and addition of constants, we know $u_{+,\xi}(t, x; \varepsilon) := a(\xi) + Bt + Ag(|x - \xi|) + \varepsilon$ is a supersolution of (2.1) and $u_{-,\xi}(t, x; \varepsilon) := a(\xi) - Bt - Ag(|x - \xi|) - \varepsilon$ is a subsolution of (2.1) for each $\varepsilon > 0$ and $\xi \in \boldsymbol{R}^N$, where g, A, B are appeared in (F4)$_+$ and (F4)$_-$, respectively.

Up to now we only consider the equation (2.1). We shall construct a supersolution and a subsolution of (2.1)-(1.2), respectively. We shall explain how to construct a supersolution of (2.1) satisfying the initial data. This is only new parts compared with [OS] because $a(x)$ is not bounded. We can construct a subsolution by similar procedure.

Lemma 3.8. *[O, Lemma 3.7] Suppose that $a(x)$ is a given uniformly continuous function on $\boldsymbol{R}^N$ (in short $a(x) \in UC(\boldsymbol{R}^N)$). For all $\varepsilon > 0$ with $0 < \varepsilon < 1$ and for each $\xi \in \boldsymbol{R}^N$, there exist $A(\varepsilon) > 0$ and $B(\varepsilon) > 0$ such that*

$$u_{+,\xi}(0, x; \varepsilon) \geq a(x) \quad \textit{for all} \quad x \in \boldsymbol{R}^N \tag{3.3}$$

and

$$\inf_{\xi \in \boldsymbol{R}^N} u_{+,\xi}(0, x; \varepsilon) \leq a(x) + \varepsilon \quad \textit{for all} \quad x \in \boldsymbol{R}^N. \tag{3.4}$$

For the completeness we shall try to prove.

Proof. It is easy to show (3.4). We put $x = \xi$ in the left side of (3.4) and observe that

$$\inf_{\xi\in\boldsymbol{R}^N} u_{+,\xi}(0,x;\varepsilon) = \inf_{\xi\in\boldsymbol{R}^N} a(\xi) + \varepsilon \leq a(x) + \varepsilon.$$

To prove the inequality (3.3) we have to show the existence of $A(\varepsilon)$ such that

$$|a(x) - a(\xi)| \leq A(\varepsilon)g(|x-\xi|) + \varepsilon. \tag{3.5}$$

Since $a(x) \in UC(\boldsymbol{R}^N)$, there exist a concave modulus function m (i.e., $m\colon [0,\infty) \to [0,\infty)$ is continuous, nondecreasing and $m(0) = 0$) such that

$$|a(x) - a(y)| \leq m(|x-y|) \quad \text{for all} \quad x, y \in \boldsymbol{R}^N.$$

Since m is concave, for each $\varepsilon > 0$ there exists a constant $M(\varepsilon) > 0$ such that

$$m(r) \leq M(\varepsilon)r + \varepsilon/2 \quad \text{for all} \quad r \in [0,\infty).$$

Then we take $A(\varepsilon)$ so that

$$M(\varepsilon)r + \varepsilon/2 \leq A(\varepsilon)g(r) + \varepsilon \quad \text{for all} \quad r \in [0,\infty).$$

Thus we obtain (3.5) which yields the inequality (3.3). □

We can prove the following by a similar argument.

Lemma 3.9. *[O, Lemma 3.8] Suppose that $a(x)$ is a given uniformly continuous function on $\boldsymbol{R}^N$ (in short $a(x) \in UC(\boldsymbol{R}^N)$). For all $\varepsilon > 0$ with $0 < \varepsilon < 1$ and for each $\xi \in \boldsymbol{R}^N$, there exist $A(\varepsilon) > 0$ and $B(\varepsilon) > 0$ such that*

$$u_{-,\xi}(0,x;\varepsilon) \leq a(x) \quad \textit{for all} \quad x \in \boldsymbol{R}^N \tag{3.6}$$

and

$$\sup_{\xi\in\boldsymbol{R}^N} u_{-,\xi}(0,x;\varepsilon) \geq a(x) - \varepsilon \quad \textit{for all} \quad x \in \boldsymbol{R}^N. \tag{3.7}$$

Now by Proposition 3.1 we conclude

Lemma 3.10. *[OS, Lemma 4.7]. Assume that F satisfies (F1), (F2) and (F3). Suppose that $a(x) \in UC(\boldsymbol{R}^N)$. Then for all $T > 0$, there exist $U_+, U_- : [0,T) \times \boldsymbol{R}^N \to \boldsymbol{R}$ such that U_+ is a supersolution of (2.1)-(1.2), U_- is a subsolution of (2.1)-(1.2) and $(U_+)_*(0,x) = (U_-)^*(0,x) = a(x)$. Moreover, $U_+(t,x) \geq U_-(t,x)$ in $[0,T) \times \boldsymbol{R}^N$.*

Sketch of proof. By Proposition 3.1

$$U_+(t,x) := \inf\{u_{+,\xi}(t,x;\varepsilon); 0 < \varepsilon < 1, \xi \in \boldsymbol{R}^N\} \tag{3.8}$$

is also a supersolution of (2.1). Applying Lemma 3.7 we observe that $U_+(0,x) = a(x)$ for all $x \in \boldsymbol{R}^N$. Moreover, since $a(x) \leq (U_+)_*(0,x) \leq U_+(0,x) = a(x)$, we see $(U_+)_*(0,x) = a(x)$. For a subsolution we set

$$U_-(t,x) := \sup\{u_{-,\xi}(t,x;\varepsilon); 0 < \varepsilon < 1, \xi \in \boldsymbol{R}^N\}. \tag{3.9}$$

By the definition of U_+ and U_-, we see $U_+(t,x) \geq U_+(0,x) = a(x) = U_-(0,x) \geq U_-(t,x)$ in $[0,T) \times \boldsymbol{R}^N$. □

Thus we constructed a supersolution and a subsolution of (1.1)-(1.2), respectively.
(d) To construct a solution of (2.1)-(1.2) we have to check that the supersolution U_+ and the subsolution U_-, respectively, fulfills the hypotheses of Proposition 3.2. The uniqueness of solutions of (2.1)-(1.2) comes from the Comparison theorem. So we shall check the condition (2.5) to U_+ and U_- in Lemma 3.10.

Lemma 3.11. *[OS, Lemma 4.8] Assume that F satisfies (F1), (F2) and (F3). Suppose that $a(x) \in UC(\boldsymbol{R}^N)$. Let U_+ and U_- be as in Lemma 3.10. Then there is a modulus function such that*

$$U_+(t,x) - U_-(0,y) \leq \omega(|x-y|+t) \quad \textit{for all} \quad t \in [0,T], x,y \in \boldsymbol{R}^N \tag{3.10}$$

and

$$U_+(0,x) - U_-(s,y) \leq \omega(|x-y|+s) \quad \textit{for all} \quad s \in [0,T], x,y \in \boldsymbol{R}^N. \tag{3.11}$$

Moreover, U_+ is local bounded from above and U_- is local bounded from bellow in $[0,T] \times \boldsymbol{R}^N$.

Note that the inequality (3.10) and (3.11) imply that U_+ and U_- fulfills (2.5).
(e) Finally, by Proposition 3.2 we can construct a solution of (1.1)-(1.2). Moreover, by the Comparison theorem we conclude

Theorem 3.12. *(Unique existence theorem) Suppose that F satisfies (F1), (F2) and (F3). Assume that $a(x) \in UC(\boldsymbol{R}^N)$. Then there exists a (unique) viscosity solution $u \in UC([0,T) \times \boldsymbol{R}^N)$ of (1.1)-(1.2).*

As a corollary we can obtain unique existence theorem for the p-Laplacian diffusion equation with $p > 1$.

Corollary 3.13. *Assume that $a(x) \in UC(\boldsymbol{R}^N)$. Then there exists a (unique) viscosity solution $u \in UC([0,T) \times \boldsymbol{R}^N)$ of (1.5)-(1.2) with $p' \geq 1$ and $p > 1$.*

Remark 3.14. Recently, the consistency of weak solutions is discussed by Juutinen, Lindqvist and Manfredi [JLM]. They study the equivalence of viscosity solutions and usual weak solutions for minus p-Laplace equation. They prove the comparison principle of viscosity solutions for minus p-Laplace equation with $p > 1$. Then the equivalence was proved. In the same way they prove the equivalence for the p-Laplace diffusion equation with $p > 1$. On the other hands, Giga [Gi] study the consistency of usual viscosity solutions (c.f [CIL]) and viscosity solutions with *admissible test functions.* For example, the comparison principle was established for the level set equation of the mean curvature flow equation by [CGG], [ES], [IS] and [OS]. In [CGG] and [ES] it was proved by using usual viscosity solutions. In [IS] and [OS] it was proved by viscosity solutions with *admissible test functions.* Giga's result is that if both solutions are exist, both solutions are same.

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